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1976 J. Phys. A: Math. Gen. 9 581

(http://iopscience.iop.org/0305-4470/9/4/015)

J. Phys. A: Math. Gen., Vol. 9. No. 4, 1976. Printed in Great Britain. © 1976

Staggered eight-vertex model on the Kagomé lattice

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Received 22 August 1975

Abstract. An eight-vertex model on the Kagomé lattice with staggered (site-dependent) vertex weights is considered. The soluble case of a free-fermion model is solved by the Pfaffian method. The staggered free-fermion model may exhibit up to five phase transitions. In general the specific heat has logarithmic singularities, except in some special cases where the system exhibits first- or second-order phase transition(s).

1. Introduction

The eight-vertex model on the square lattice was solved by Baxter (1971). Wegner (1972) pointed out that the Ashkin–Teller model (Ashkin and Teller 1943) on the square lattice is equivalent to a special case of the staggered eight-vertex model on the square lattice. Wu (1975, private communication) showed that the triangular Ashkin–Teller model (Enting 1975) is equivalent to a special case of the staggered eight-vertex model on the Kagomé lattice. The Pfaffian solutions of the staggered ice-rule vertex model on the square and Kagomé lattices have been obtained respectively by Wu and Lin (1975) and by Lin (1975). Recently Hsue *et al* (1975) considered the general staggered eight-vertex model on the square lattice. They discussed in detail the soluble case of a free-fermion model where the system may exhibit up to three phase transitions. In general the specific heat has logarithmic singularities, except in special cases it diverges with an exponent $\alpha = \frac{1}{2}$ above the unique transition temperature T_c and the system is frozen below T_c.

The motivation for this paper is to generalize the work of Hsue, Lin and Wu to the Kagomé lattice. The staggered eight-vertex model on the Kagomé lattice is described in § 2. Symmetry relations are discussed in § 3. When the vertices satisfy the free-fermion condition, the model can be solved by the Pfaffian method (Montroll 1964). The Pfaffian solution is given in § 4. There are four cases where the free-fermion condition is satisfied at all temperatures. These cases are examined in detail in § 5. Our conclusion is given in § 6.

2. Definition of the model

Place arrows on the bonds of a Kagomé lattice L of N sites and allow only those configurations with an even number of arrows pointing into each vertex. The three sublattices of L are denoted by A, B and C, as shown in figure 1. The eight possible

† Supported in part by the National Science Council, Republic of China.

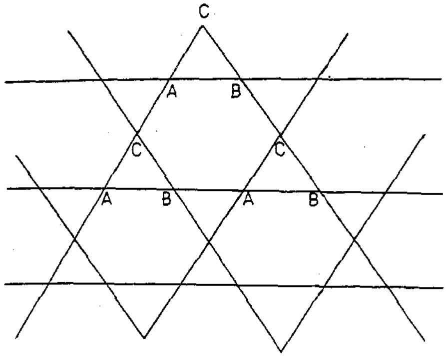

Figure 1. The Kagomé lattice with three sublattices A, B and C.

configurations allowed at each vertex are shown in figure 2, where each vertex type is assigned a weight. Let the vertex weights be

$$
\begin{aligned}
\{\omega\} &= \{\omega_1, \omega_2, \ldots, \omega_8\} && \text{on A} \\
\{\omega'\} &= \{\omega'_1, \omega'_2, \ldots, \omega'_8\} && \text{on B} \\
\{\omega''\} &= \{\omega''_1, \omega''_2, \ldots, \omega''_8\} && \text{on C.}
\end{aligned}
\tag{1}
$$

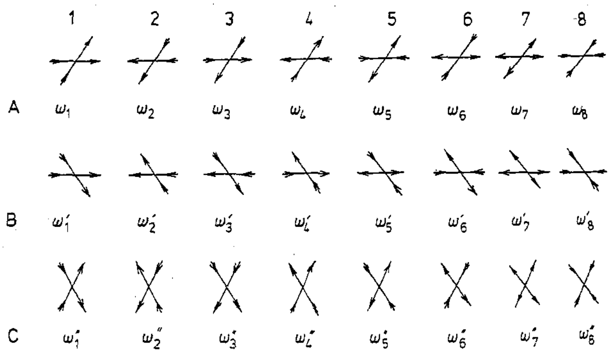

Figure 2. The eight-vertex configurations and the associated weights.

The partition function is

$$Z = \Sigma(\Pi\omega_i^{n_i})(\Pi\omega_i'^{n'_i})(\Pi\omega_i''^{n''_i}) \tag{2}$$

where the summation is extended to all allowed arrow configurations on L, and $n_i(n'_i, n''_i)$ is the number of ith-type sites on A(B, C). The goal is to compute the 'free energy'

$$\psi = \lim_{N\to\infty} \frac{1}{N} \ln Z. \tag{3}$$

In ferroelectric language the vertex weights are the Boltzmann factors

$$\omega_i = \exp(-\beta e_i) \qquad \omega'_i = \exp(-\beta e'_i) \qquad \omega''_i = \exp(-\beta e''_i) \tag{4}$$

where $\beta = 1/kT$, k is the Boltzmann constant, T is the temperature, and e_i, e'_i, e''_i are the vertex energies.

When $\omega_7 = \omega_8 = \omega'_7 = \omega'_8 = \omega''_7 = \omega''_8 = 0$, this model reduces to the ice-rule case considered before by Lin (1975).

3. Symmetry relations

The partition function Z possesses some symmetry relations which follow from general considerations. We write

$$Z = Z(12345678, 1'2'3'4'5'6'7'8', 1''2''3''4''5''6''7''8'') \tag{5}$$

where i, i', i'' denote respectively ω_i, ω'_i, ω''_i. Reversing all arrows along one of the three directions in the Kagomé lattice, we obtain

$$\begin{aligned} Z &= Z(43217856, 3'4'1'2'8'7'6'5', 1''2''3''4''5''6''7''8'') \\ &= Z(34128765, 1'2'3'4'5'6'7'8', 3''4''1''2''8''7''6''5'') \\ &= Z(12345678, 4'3'2'1'7'8'5'6', 4''3''2''1''7''8''5''6''). \end{aligned} \tag{6}$$

Reversing all arrows implies

$$Z = Z(21436587, 2'1'4'3'6'5'8'7', 2''1''4''3''6''5''8''7''). \tag{7}$$

Reflection symmetry implies

$$\begin{aligned} Z &= Z(1'2'4'3'6'5'7'8', 12436578, 1''2''4''3''6''5''7''8'') \\ &= Z(2'1'3'4'6'5'7'8', 21346578, 2''1''3''4''6''5''7''8'') \\ &= Z(21346578, 3''4''1''2''5''6''7''8'', 3'4'1'2'5'6'7'8') \\ &= Z(12436578, 4''3''2''1''5''6''7''8'', 4'3'2'1'5'6'7'8') \\ &= Z(4''3''2''1''5''6''7''8'', 2'1'3'4'6'5'7'8', 43215678) \\ &= Z(3''4''1''2''5''6''7''8'', 1'2'4'3'6'5'7'8', 34125678). \end{aligned} \tag{8}$$

Rotation symmetry leads to

$$\begin{aligned} Z &= Z(2'1'4'3'5'6'7'8', 4''3''1''2''5''6''7''8'', 43125678) \\ &= Z(3''4''2''1''6''5''7''8'', 21436578, 3'4'2'1'5'6'7'8'). \end{aligned} \tag{9}$$

Finally there is the weak-graph symmetry (Nagle and Temperley 1968) which is a local property of a lattice and is valid even if the weights are site dependent.

4. Pfaffian solution

A vertex model can be solved by the Pfaffian method if the free-fermion condition is satisfied at each vertex (Fan and Wu 1970). In our model, the condition reads

$$\begin{aligned} &\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8 \\ &\omega'_1\omega'_2 + \omega'_3\omega'_4 = \omega'_5\omega'_6 + \omega'_7\omega'_8 \\ &\omega''_1\omega''_2 + \omega''_3\omega''_4 = \omega''_5\omega''_6 + \omega''_7\omega''_8. \end{aligned} \tag{10}$$

Under this condition the partition function is equal to a Pfaffian which is evaluated in the appendix. The result is

$$\psi=\frac{1}{16\pi^2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\ \ln F(\theta,\phi) \tag{11}$$

where

$$\begin{aligned}F(\theta,\phi)=&\ \Omega_1^2+\Omega_2^2+\Omega_3^2+\Omega_4^2-2(\Omega_1\Omega_3-\Omega_2\Omega_4)\cos\theta-2(\Omega_1\Omega_4-\Omega_2\Omega_3)\cos(\theta+\phi)\\&+2(\Omega_3\Omega_4-\Omega_1\Omega_2)\cos\phi-4a\sin\theta\sin(\theta+\phi)-4b\sin\phi\sin(\theta+\phi)\\&-4c\sin\theta\sin\phi-4d\sin^2\theta-4e\sin^2\phi-4f\sin^2(\theta+\phi)\end{aligned}$$

$$\begin{aligned}\Omega_1&=\omega_1\omega_1'\omega_1''+\omega_2\omega_2'\omega_2''+\omega_7\omega_8'\omega_6''+\omega_8\omega_7'\omega_5''\\ \Omega_2&=\omega_3\omega_1'\omega_3''+\omega_4\omega_2'\omega_4''+\omega_5\omega_7'\omega_8''+\omega_6\omega_8'\omega_7''\\ \Omega_3&=\omega_1\omega_4'\omega_4''+\omega_2\omega_3'\omega_3''+\omega_7\omega_6'\omega_8''+\omega_8\omega_5'\omega_7''\\ \Omega_4&=\omega_3\omega_4'\omega_2''+\omega_4\omega_3'\omega_1''+\omega_5\omega_5'\omega_6''+\omega_6\omega_6'\omega_5''\\ a&=(\omega_1'\omega_2'-\omega_7'\omega_8')(\omega_6\omega_8\omega_5''\omega_7''+\omega_5\omega_7\omega_6''\omega_8''-\omega_1\omega_4\omega_1''\omega_4''-\omega_2\omega_3\omega_2''\omega_3'')\\ b&=(\omega_1\omega_2-\omega_7\omega_8)(\omega_5'\omega_8'\omega_6''\omega_7''+\omega_6'\omega_7'\omega_5''\omega_8''-\omega_1'\omega_3'\omega_1''\omega_3''-\omega_2'\omega_4'\omega_2''\omega_4'')\\ c&=(\omega_3''\omega_4''-\omega_7''\omega_8'')(\omega_6\omega_7\omega_6'\omega_8'+\omega_5\omega_8\omega_5'\omega_7'-\omega_2\omega_4\omega_2'\omega_3'-\omega_1\omega_3\omega_1'\omega_4')\\ d&=(\omega_3\omega_4-\omega_7\omega_8)(\omega_1'\omega_2'-\omega_7'\omega_8')(\omega_3''\omega_4''-\omega_7''\omega_8'')\\ e&=(\omega_1\omega_2-\omega_7\omega_8)(\omega_3'\omega_4'-\omega_7'\omega_8')(\omega_3''\omega_4''-\omega_7''\omega_8'')\\ f&=(\omega_1\omega_2-\omega_7\omega_8)(\omega_1'\omega_2'-\omega_7'\omega_8')(\omega_1''\omega_2''-\omega_7''\omega_8'').\end{aligned} \tag{12}$$

Although there are 21 independent vertex weights to start with, the final expression contains only 10 independent parameters. It can be shown that

$$\begin{aligned}&\Omega_3\Omega_4\geqslant a\geqslant-\Omega_1\Omega_2, &&\Omega_2\Omega_4\geqslant b\geqslant-\Omega_1\Omega_3\\ &\Omega_1\Omega_4\geqslant c\geqslant-\Omega_2\Omega_3, &&\Omega_1^2+\Omega_2^2+\Omega_3^2+\Omega_4^2\geqslant 8\max\{d,e,f\}.\end{aligned} \tag{13}$$

The expression for the free energy is the same as that for the free energy of the 32-vertex free-fermion model on a triangular lattice (Sacco and Wu 1975) except that each Ω_i is the sum of four terms instead of two.

It is easy to check that $F(\theta,\phi)=0$ at the following points:

$$\begin{aligned}&\theta=\phi=0 &&\Omega_1=\Omega_2+\Omega_3+\Omega_4\\ &\theta=\pi,\ \phi=0 &&\Omega_2=\Omega_1+\Omega_3+\Omega_4\\ &\theta=0,\ \phi=\pi &&\Omega_3=\Omega_1+\Omega_2+\Omega_4\\ &\theta=\phi=\pi &&\Omega_4=\Omega_1+\Omega_2+\Omega_3.\end{aligned} \tag{14}$$

In physical models, the vertex weights ω_i, ω_i', ω_i'' are the Boltzmann factors $\exp(-\beta e_i)$, $\exp(-\beta e_i')$, $\exp(-\beta e_i'')$. In general, all the zeros of $F(\theta,\phi)$ are given by (14) and the critical temperature T_c is determined by $\Delta(T_c)=0$ where

$$\Delta(T)=\Omega_1+\Omega_2+\Omega_3+\Omega_4-2\max\{\Omega_1,\Omega_2,\ \Omega_3,\Omega_4\}. \tag{15}$$

Since neither ψ nor its derivatives can be expressed in terms of known functions except in special cases, we shall confine our discussion to the critical behaviour of ψ. To be specific, we consider the non-analyticity of ψ at $\Omega_1 = \Omega_2 + \Omega_3 + \Omega_4$. Following Hsue *et al* (1975), we expand $F(\theta, \phi) = 0$ about $\theta = \phi = 0$ and obtain

$$\psi_{\text{singular}} \sim \int d\theta \int d\phi \ln[(\Omega_1 - \Omega_2 - \Omega_3 - \Omega_4)^2 + \alpha\theta^2 + \beta\theta\phi + \gamma\phi^2] \tag{16}$$

where only the lower integration limits are needed. The above integration can be performed by first diagonalizing the quadratic form in θ and ϕ. One finds (Hsue *et al* 1975)

$$\psi_{\text{singular}} \sim (T - T_c)^2 \ln|T - T_c| \tag{17}$$

which leads to the Ising behaviour. The specific heat diverges logarithmically. The argument breaks down if

$$\beta^2 = 4\alpha\gamma \tag{18}$$

at T_c (Hsue *et al* 1975). The condition (18) implies that there exist zeros of $F(\theta, \phi) = 0$ which are not given by (14). The critical behaviour of ψ in this special case will be discussed in the following section.

5. Exactly soluble models

The free-fermion condition (10) can be satisfied at all temperatures provided the vertex energies e_i, e'_i, e''_i satisfy some identities. There are four exactly soluble models (others are related to them by symmetry):

(1)
$$\begin{aligned} &e_1 + e_2 = e_7 + e_8 && e_3 + e_4 = e_5 + e_6 \\ &e'_1 + e'_2 = e'_7 + e'_8 && e'_3 + e'_4 = e'_5 + e'_6 \\ &e''_1 + e''_2 = e''_5 + e''_6 && e''_3 + e''_4 = e''_7 + e''_8 \end{aligned} \tag{19}$$

(2)
$$\begin{aligned} &e_1 + e_2 = e_7 + e_8 && e_3 + e_4 = e_5 + e_6 \\ &e'_1 + e'_2 = e'_5 + e'_6 && e'_3 + e'_4 = e'_7 + e'_8 \\ &e''_1 + e''_2 = e''_5 + e''_6 && e''_3 + e''_4 = e''_7 + e''_8 \end{aligned} \tag{20}$$

(3)
$$\begin{aligned} &e_1 + e_2 = e_5 + e_6 && e_3 + e_4 = e_7 + e_8 \\ &e'_1 + e'_2 = e'_5 + e'_6 && e'_3 + e'_4 = e'_7 + e'_8 \\ &e''_1 + e''_2 = e''_7 + e''_8 && e''_3 + e''_4 = e''_5 + e''_6 \end{aligned} \tag{21}$$

(4)
$$\begin{aligned} &e_1 + e_2 = e_7 + e_8 && e_3 + e_4 = e_5 + e_6 \\ &e'_1 + e'_2 = e'_5 + e'_6 && e'_3 + e'_4 = e'_7 + e'_8 \\ &e''_1 + e''_2 = e''_7 + e''_8 && e''_3 + e''_4 = e''_5 + e''_6. \end{aligned} \tag{22}$$

5.1. *Model 1*

Equations (19) imply $a = b = c = d = e = f = 0$. This model is identical to the free-fermion model of Fan and Wu (1970) except that each Ω_i is the summation of four Boltzmann factors instead of one. The critical condition is $\Delta(T_c) = 0$. In the model of

Fan and Wu, $\Delta = 0$ has one solution at $T = T_c$ and $\Delta(T) > (<) 0$ if and only if $T > (<) T_c$. In our model, $\Delta(T_c) = 0$ may have up to five solutions.

If $\Omega_1\Omega_2\Omega_3\Omega_4 \neq 0$, then ψ cannot be evaluated in a closed form but the first derivative can be expressed in terms of the complete elliptical integrals of the first and third kinds (Green and Hurst 1964). Consequently the specific heat has a logarithmic divergence at each transition temperature. The system may exhibit up to five phase transitions. For example, if $e_2 = e_7 = e_8 = \infty$, $e_1 = e_i' = e_1'' = e_2'' = e_5'' = e_6'' = 0$, $e_3 = e_4 = e_5 = e_6 = \epsilon > 0$, $e_3'' = e_8'' = -0{\cdot}9\epsilon$, $e_4'' = e_7'' = 20\epsilon$, then we have

$$\Omega_1 = 1 \qquad \Omega_2 = 2(e^{-0\cdot 1\beta\epsilon} + e^{-21\beta\epsilon})$$
$$\Omega_3 = e^{-20\beta\epsilon} \qquad \Omega_4 = 4\,e^{-\beta\epsilon} \tag{23}$$

and the critical condition implies

$$\Omega_1 = \Omega_2 + \Omega_3 + \Omega_4 \qquad \text{at } T = T_c^1$$
$$\Omega_2 = \Omega_1 + \Omega_3 + \Omega_4 \qquad \text{at } T = T_c^2,\ T_c^3$$
$$\Omega_4 = \Omega_1 + \Omega_2 + \Omega_3 \qquad \text{at } T = T_c^4,\ T_c^5 \tag{24}$$

where $T_c^i > T_c^j$ if $i > j$.

If one of Ω_i vanishes, this model reduces to the modified KDP model of Wu (1968). A second-order phase transition occurs at each transition temperature determined by $\Delta(T_c) = 0$. The specific heat behaves as $\Delta^{-1/2}$ when Δ approaches zero from above. The system may exhibit up to four phase transitions. For example, if $e_1 = e_7 = e_2' = e_7' = \infty$, $e_3 = e_4 = e_5 = e_6 = e_1' = e_i'' = 0$, $e_3' = e_4' = e_5' = e_6' = 10\epsilon > 0$, $e_8' = 40\epsilon$, $e_2 = e_8 = -9\epsilon$, then we have

$$\Omega_1 = 0 \qquad \Omega_2 = 1 + e^{-40\beta\epsilon}$$
$$\Omega_3 = 2\,e^{-\beta\epsilon} \qquad \Omega_4 = 4\,e^{-10\beta\epsilon} \tag{25}$$

and $\Delta = 0$ implies

$$\Omega_2 = \Omega_3 + \Omega_4 \qquad \text{at } T = T_c^1$$
$$\Omega_3 = \Omega_2 + \Omega_4 \qquad \text{at } T = T_c^2,\ T_c^3 \tag{26}$$
$$\Omega_4 = \Omega_2 + \Omega_3 \qquad \text{at } T = T_c^4$$

where $T_c^i > T_c^j$ if $i > j$. In this case we have (Wu 1968)

$$\psi = \begin{cases} \frac{1}{2}\ln\Omega_2 & T \leqslant T_c^1 \\ \frac{1}{2}\ln\Omega_3 & T_c^3 \geqslant T \geqslant T_c^2 \\ \frac{1}{2}\ln\Omega_4 & T \geqslant T_c^4 \\ \frac{1}{2}\ln\Omega_4 + \frac{1}{8\pi}\int_{-\phi_1}^{\phi_1} \ln\left(\frac{\Omega_2^2 + \Omega_3^2 + 2\Omega_2\Omega_3\cos\phi}{\Omega_4^2}\right) d\phi & \text{otherwise} \end{cases} \tag{27}$$

where

$$\cos\phi_1 = (\Omega_4^2 - \Omega_2^2 - \Omega_3^2)/2\Omega_2\Omega_3. \tag{28}$$

The specific heat diverges with an exponent† $\alpha = \frac{1}{2}$ at $T = T_c^1,\ T_c^3$ and $\alpha' = \frac{1}{2}$ at $T = T_c^2$,

† We use the standard definitions of critical-point exponents (Stanley 1971).

T_c^A. It is interesting to note that the specific heat vanishes *above* the critical temperature T_c^A in this particular example.

Finally we consider the situation where two of Ω_i vanish. For example, if $e_1 = e_2 = e_7 = e_8 = \infty$ then we have $\Omega_1 = \Omega_3 = 0$ and

$$\psi = \tfrac{1}{4} \ln \max\{\Omega_2, \Omega_4\}, \tag{29}$$

where the first derivative of ψ has a jump discontinuity at $\Omega_2 = \Omega_4$ (first-order phase transition). The critical condition $\Omega_2 = \Omega_4$ may have up to two solutions. For example, if $e_3'' = e_7'' = \infty$, $e_4 = e_5 = e_6 = 10\epsilon > 0$, $e_4'' = e_8'' = -9\epsilon$, $e_3 = e_i' = e_1'' = e_2'' = e_5'' = e_6'' = 0$, then we have

$$\Omega_2 = 2\,\mathrm{e}^{-\beta\epsilon} \qquad \Omega_4 = 1 + 3\,\mathrm{e}^{-10\beta\epsilon} \tag{30}$$

and $\Omega_2 = \Omega_4$ has two solutions.

5.2. *Model 2*

Equations (20) imply $b = c = d = e = f = 0$; we define

$$\Omega_5\Omega_6 = \Omega_3\Omega_4 - a, \qquad \Omega_7\Omega_8 = \Omega_1\Omega_2 + a, \qquad \alpha = \theta, \qquad \beta = \theta + \phi. \tag{31}$$

The free energy is now given by

$$\psi = \frac{1}{16\pi^2} \int_0^{2\pi} \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta \ln F_0(\alpha, \beta) \tag{32}$$

where

$$F_0 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2 - 2(\Omega_1\Omega_3 - \Omega_2\Omega_4)\cos\alpha - 2(\Omega_1\Omega_4 - \Omega_2\Omega_3)\cos\beta$$
$$+ 2(\Omega_3\Omega_4 - \Omega_7\Omega_8)\cos(\alpha - \beta) + 2(\Omega_3\Omega_4 - \Omega_5\Omega_6)\cos(\alpha + \beta). \tag{33}$$

Note that ψ is now exactly of the form of the free energy for a uniform free-fermion model (Fan and Wu 1970, equation (16)). The integral (32) has been investigated by Hsue *et al* (1975). They found that the critical condition is given by $\Delta(T_c) = 0$.

If $\Omega_5\Omega_6\Omega_7\Omega_8 \neq 0$, then ψ cannot be evaluated in a closed form but its first derivative is given by the complete elliptical integrals of the first and third kinds (Green and Hurst 1964). Therefore the specific heat has a logarithmic divergence at each transition temperature. Since $\Delta(T_c) = 0$ may have as much as five solutions, the system may exhibit up to five phase transitions.

If $\Omega_5\Omega_6\Omega_7\Omega_8 = 0$, then we have (Hsue *et al* 1975, equations (42–4))

$$\psi = \begin{cases} \tfrac{1}{2} \ln \max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\} & \Delta \leqslant 0 \\ \tfrac{1}{2} \ln \max\{\Omega_1, \Omega_4\} & \\ \quad + \dfrac{1}{8\pi} \displaystyle\int_{-\phi_1}^{\phi_1} \ln\left(\frac{\Omega_2^2 + \Omega_3^2 + 2\Omega_2\Omega_3\cos\phi}{\Omega_1^2 + \Omega_4^2 - 2\Omega_1\Omega_4\cos\phi}\right) \mathrm{d}\phi & \Delta \geqslant 0 \end{cases} \tag{34}$$

where

$$\cos\phi_1 = (\Omega_1^2 + \Omega_4^2 - \Omega_2^2 - \Omega_3^2)/2(\Omega_1\Omega_4 + \Omega_2\Omega_3). \tag{35}$$

The specific heat behaves as $\Delta^{-1/2}$ when $\Delta \to 0^+$. The system may exhibit up to four phase transitions.

In the special case where $\Omega_5\Omega_6\Omega_7\Omega_8=0$ and two of Ω_i vanish, first-order phase transition occurs. For example, if $e_1=e_2=e_7=e_8=\infty$, then we have $\psi=\frac{1}{4}\ln\max\{\Omega_2, \Omega_4\}$ and the first derivative of ψ has a jump discontinuity at $\Omega_2=\Omega_4$ which may have up to two solutions.

5.3. *Model* 3

Equations (21) imply $d=e=f=0$. The case of $a=b=c=0$ reduces to model 1. The cases of ($a\neq 0$, $b=c=0$), ($b\neq 0$, $a=c=0$), ($c\neq 0$, $a=b=0$) reduce to model 2. Otherwise neither ψ nor its derivatives can be expressed in terms of known functions. Following the argument of Hsue *et al* (1975), we have

$$\psi_{\text{singular}}\sim t^2\ln|t| \qquad t=(T-T_c)/T_c\to 0. \tag{36}$$

The critical condition is $\Delta(T_c)=0$ which may have up to five solutions.

5.4. *Model* 4

Equations (22) imply $b=e=f=0$. In the general case where none of a, c, d vanish, neither ψ nor its derivatives can be expressed in terms of known functions. Following the argument of Hsue *et al* (1975), the critical condition is $\Delta(T_c)=0$ and the singular behaviour of ψ is given by (36). The case of $a=c=d=0$ reduces to model 1. The case of $a\neq 0$, $c=d=0$ (or $c\neq 0$, $a=d=0$) reduces to model 2. The case of $d=0$, $a\neq 0$, $c\neq 0$ reduces to model 3.

The case of $a=c=0$, $d\neq 0$ implies

$$\begin{array}{llll} e_1=e_8 & e_2=e_7 & e_3=e_5 & e_4=e_6 \\ e_1'=e_5' & e_2'=e_6' & e_3'=e_8' & e_4'=e_7' \\ e_1''=e_7'' & e_2''=e_8'' & e_3''=e_6'' & e_4''=e_5''. \end{array} \tag{37}$$

It follows from (37) that $\Omega_1=\Omega_3$, $\Omega_2=\Omega_4$ and

$$F(\theta,\phi)=2(\Omega_1^2+\Omega_2^2)-2(\Omega_1^2-\Omega_2^2)\cos\theta-4d\sin^2\theta. \tag{38}$$

It can be shown that the expression (38) is always greater than zero and therefore ψ has no singularity.

The case of $c=0$, $a\neq 0$, $d\neq 0$ implies

$$\begin{array}{llll} e_1=e_8 & e_2=e_7 & e_3=e_5 & e_4=e_6 \\ e_1'=e_5' & e_2'=e_6' & e_3'=e_8' & e_4'=e_7'. \end{array} \tag{39}$$

The free energy can be written in the form

$$\psi=\frac{1}{16\pi^2}\int_0^{2\pi}\mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\beta\,\ln F(\alpha,\beta) \tag{40}$$

where

$$F(\alpha,\beta)=F_0(\alpha,\beta)-4d\sin^2\alpha \tag{41}$$

and F_0 is defined by (33). The integral (40) has been discussed in detail by Hsue *et al*

(1975, equation (57)). They found that the critical condition is $\Delta(T_c)=0$ and

$$\psi=\frac{1}{8\pi}\int_0^{2\pi} d\alpha \ln(A+Q^{1/2}) \tag{42}$$

where

$$A(\alpha)=\tfrac{1}{2}(\Omega_1^2+\Omega_2^2+\Omega_3^2+\Omega_4^2)-(\Omega_1\Omega_3-\Omega_2\Omega_4)\cos\alpha-2d\sin^2\alpha\geqslant 0 \tag{43}$$

$$\begin{aligned}Q(\alpha)&=[2d\sin^2\alpha+(\Omega_1\Omega_3+\Omega_2\Omega_4)\cos\alpha-\tfrac{1}{2}(\Omega_1^2-\Omega_2^2+\Omega_3^2-\Omega_4^2)]^2\\&\quad+8d\Omega_2\Omega_4(1-\cos\alpha)\sin^2\alpha+4\Delta_1\sin^2\alpha\\&=[2d\sin^2\alpha-(\Omega_1\Omega_3+\Omega_2\Omega_4)\cos\alpha+\tfrac{1}{2}(\Omega_1^2-\Omega_2^2+\Omega_3^2-\Omega_4^2)]^2\\&\quad+8d\Omega_1\Omega_3(1+\cos\alpha)\sin^2\alpha+4\Delta_2\sin^2\alpha\geqslant 0\end{aligned} \tag{44}$$

$$\Delta_1=\Omega_5\Omega_6\Omega_7\Omega_8-d(\Omega_2+\Omega_4)^2 \tag{45}$$

$$\Delta_2=\Omega_5\Omega_6\Omega_7\Omega_8-d(\Omega_1+\Omega_3)^2. \tag{46}$$

The singular part of ψ behaves as (36) except for

$$\Delta_1=\Omega_2\Omega_4=0 \qquad \text{or} \qquad \Delta_2=\Omega_1\Omega_3=0. \tag{47}$$

Under the condition (47) Q is a complete square and (Hsue *et al* 1975, equation (68))

$$\psi_{\text{singular}}\sim t^{3/2} \qquad t\to 0^+. \tag{48}$$

The condition (47) can be realized by taking, e.g.,

$$e_3=e_5=e_3'=e_8'=e_3''=e_4''=e_6''=\infty. \tag{49}$$

The case of $a=0$, $c\neq 0$, $d\neq 0$ can be treated in the same way.

6. Conclusion

We have considered the staggered free-fermion eight-vertex model on the Kagomé lattice and examined the exactly soluble cases where the vertex weights satisfy the free-fermion condition at all temperatures. We found that the critical condition is always given by $\Delta(T_c)=0$ which may have as many as five solutions. In general the specific heat has logarithmic singularities except in some special cases where the system may exhibit up to four second-order phase transitions (α or $\alpha'=\frac{1}{2}$) or up to two first-order phase transitions.

Acknowledgment

The author is indebted to Professor F Y Wu for suggesting this problem and encouragement.

Appendix. Pfaffian solution

Expand each site of L into a 'city' of four terminals to form a dimer lattice L^{Δ} whose unit cell is shown in figure 3. Following the same procedure as Hsue *et al* (1975), we have

$$\psi = \frac{1}{16\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \, \ln[(\omega_2\omega_2'\omega_2'')^2 D(\theta, \phi)] \tag{A.1}$$

where $D(\theta, \phi) =$

$$\left|\begin{array}{cccc:cccc:cccc}
0 & u_3 & -u_8 & -u_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-u_3 & 0 & u_6 & -u_7 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{-i(\theta+\phi)} & 0 \\
u_8 & -u_6 & 0 & u_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
u_5 & u_7 & -u_4 & 0 & 0 & 0 & -e^{-i\phi} & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & u_3'' & -u_8'' & -u_5'' & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -u_3'' & 0 & u_6'' & -u_7'' & -e^{-i\theta} & 0 & 0 & 0 \\
0 & 0 & 0 & e^{i\phi} & u_8'' & -u_6'' & 0 & u_4'' & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & u_5'' & u_7'' & -u_4'' & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & e^{i\theta} & 0 & 0 & 0 & u_3' & -u_8' & -u_5' \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -u_3' & 0 & u_6' & -u_7' \\
0 & e^{i(\theta+\phi)} & 0 & 0 & 0 & 0 & 0 & 0 & u_8' & -u_6' & 0 & u_4' \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_5' & u_7' & -u_4' & 0
\end{array}\right|$$

$$u_i = \omega_i/\omega_2 \qquad u_i' = \omega_i'/\omega_2' \qquad u_i'' = \omega_i''/\omega_2''$$

Equation (A.1) reduces to equation (6) after some algebra.

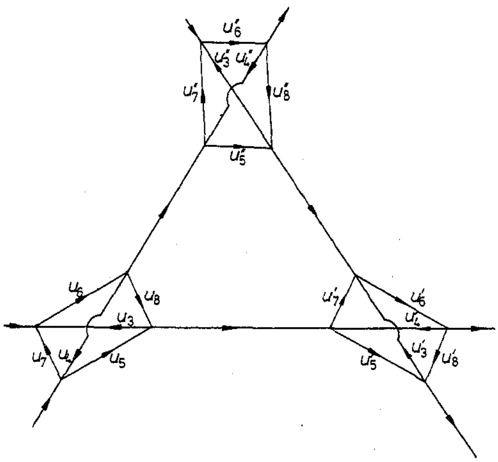

Figure 3. A unit cell of the dimer lattice L^{Δ}.

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